A variational calculation of particle-antiparticle bound states in the scalar Yukawa model

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Abstract

We consider particle-antiparticle bound states in the scalar Yukawa (Wick-Cutkosky) model. The variational method in the Hamiltonian formalism of quantum field theory is employed. A reformulation of the model is studied, in which covariant Green's functions are used to solve for the mediating field in terms of the particle fields. A simple Fock-state variational ansatz is used to derive a relativistic equation for the particle-antiparticle states. This equation contains one-quantum-exchange and virtual-annihilation interactions. It is shown that analytic solutions of this equation can be obtained for the simplified case where only the virtual annihilation interaction is retained. More generally, numerical and perturbative solutions of the equation are obtained for the massive and massless-exchange cases. We compare our results with various Bethe-Salpeter-based calculations.

1. Introduction

The variational method has been used sparingly in treating few-particle bound and quasi-bound states in quantum field theory. Yet, it is in principle appealing, particularly for strongly coupled systems, because of its largely analytic and non-perturbative nature. The early papers that kindled interest in the variational method in QFT are those of Schiff [1], Coleman and Weinberg [2,3], Jackiw [4] Barnes and Ghandour [5], Stevenson [6], Tarrach [7] and others [8]. A brief review of this approach to few-body bound states in the Hamiltonian formalism of QFT up to 1996 is given in ref. [9].

The variational method is, of course, only as good as the trial states that are being employed. In the functional formulation (e.g. [5,8]) one is restricted largely to wave-functionals of the Gaussian type, due to the difficulty of handling analytically non-Gaussian functional integrals [10]. Another approach is to expand the trial state in a Fock-space basis. Indeed, the early work of Tamm [11] and Dancoff [12] is along these lines, though it was not variationally formulated.

The Hamiltonian formalism of QFT can be expressed in terms of the QFTheoretic eigenvalue equation

$$\hat{H}|\Psi\rangle = E|\Psi\rangle, \tag{1}$$

where $\hat{H}$ is the QFTheoretic hamiltonian operator, and E is the energy of the system under study. Such an equation is generally impossible to solve, except for some special models, particularly in 1+1 (one spatial coordinate plus time), such as the Thirring [13]and Schwinger [14] models. In the variational approach within the Hamiltonian formalism one seeks approximate solutions to Eq. (1) by using the variational principle

$$\delta\langle\Psi_t|\hat{H} - E|\Psi_t\rangle = 0, \tag{2}$$

where $|\Psi_t\rangle$ is a suitably chosen trial state containing adjustable features (parameters, functions). Some examples of the application of this method to bound states in scalar ϕ^6 theory are given in refs. [6c] and [15].

One of the difficulties of the Fock-state expansion approach for realistic models in 3+1 (such as the Yukawa model, QED, etc.) is that it results in an infinite system of coupled, multi-dimensional integral equations to be solved - an impossible task. Truncation, or other approximation schemes, undermine the strict variational nature of the approximation. This, then, puts into question the validity of results at strong coupling, precisely in the domain that one wishes to address in a non-perturbative approach. It has been pointed out recently [16] that various models can be reformulated in such a way that this difficulty can be circumvented.

In this paper we shall implement the approach given in reference [16] to particle-antiparticle bound states in the scalar Yukawa model, in which scalar particles interact via a mediating real (massive or massless) scalar field. The treatment of two-body bound states in this model by means of the Bethe-Salpeter equation in the ladder approximation is known as the Wick-Cutkosky model [17,18]. In addition to these original solutions of Wick and Cutkosky, the scalar Yukawa model has

been used as a prototype QFT in many studies. It has been investigated quite extensively in various formalisms, such as the light-cone formulation [19-22], various Bethe-Salpeter-based approaches [23-27], and others [28, 29, 30]. The work of Nieuwenhuis and Tjon [27b], in particular, gives a comparison of a number of quasipotential approximations. This makes the model appealing as a relatively simple test case. (We make no effort here to give an exhaustive survey of the literature on this model. Many relevant papers are cited in the references which we quote. A review of the Wick-Cutkosky model to 1988 is given by Nakanishi [31], while many mathematical details are given in the work of Silagadze [32].)

In the scalar yukawa model massive scalar particles interact via a mediating real scalar field. It is based on the Lagrangian density ($\hbar = c = 1$)

$$\begin{aligned} \mathcal{L} &= \partial^\nu \phi^*(x)\, \partial_\nu \phi(x) - m^2 \phi^*(x)\phi(x) \\ &+ \frac{1}{2}\partial^\nu \chi(x)\, \partial_\nu \chi(x) - \frac{1}{2}\mu^2 \chi^2(x) - g\phi^*(x)\phi(x)\, \chi(x) \end{aligned} \tag{3}$$

The mediating "chion" field can be massive ($\mu \neq 0$) or massless (i.e. $\mu = 0$, as in the original Wick, Cutkosky work [17, 18]). The coupling constant g has dimensions of $(\text{mass})^{\frac{5}{2}-\frac{N}{2}}$ in N+1 dimensionless. A slightly simpler model, in which ϕ is real, is often considered. In that case there are only particles and no antiparticles. These models are closely related, since the forces among particles (and/or antiparticles) are only attractive (i.e. gravity-like, rather than electromagnetic-like). Of course, in the case where ϕ is real there is no particle-antiparticle annihilation.

2. Reformulation of the model

The fields ϕ and χ of the model (3) satisfy the Euler-Lagrange equations

$$\partial^\nu \partial_\nu \chi(x) + \mu^2 \chi(x) = -g\phi^*(x)\phi(x), \tag{4}$$

$$\partial^\nu \partial_\nu \phi(x) + m^2 \phi(x) = -g\phi(x)\chi(x), \tag{5}$$

and the conjugate of (5). Equation (4) has the formal solution

$$\chi(x) = \chi_0(x) + \int dx'\, D(x - x')\, \rho(x'), \tag{6}$$

where $dx = d^N x\, dt$ in $N+1$ dimensions, $\rho(x) = -g\phi^*(x)\phi(x)$, $\chi_0(x)$ satisfies the homogeneous (or free field) equation (eq. (4) with $g = 0$), while $D(x - x')$ is a covariant Green function (or chion propagator, in QFTheoretic language), such that

$$\left(\partial^\nu \partial_\nu + \mu^2\right) D(x - x') = \delta^{N+1}(x - x'). \tag{7}$$

Equation (7) does not specify $D(x-x')$ uniquely since, for example, any solution of the homogeneous equation can be added to it without invalidating (7). Boundary conditions based on physical considerations are used to pin down the form of D.

Substitution of the formal solution (6) into eq. (5) yields the equation

$$\partial^\nu \partial_\nu \phi(x) + m^2 \phi(x) = -g\phi(x)\chi_0(x) - g\phi(x) \int dx' D(x-x')\rho(x'). \tag{8}$$

Equation (8) is derivable from the action principle $\delta \int dx\, \mathcal{L} = 0$, corresponding to the Lagrangian density

$$\begin{aligned} \mathcal{L} &= \partial^\nu \phi^*(x) \partial_\nu \phi(x) - m^2 \phi^*(x)\phi(x) - g\phi^*(x)\phi(x)\,\chi_0(x) \\ &+ \frac{1}{2} \int dx' \rho(x) D(x-x') \rho(x'), \end{aligned} \tag{9}$$

provided that $D(x-x') = D(x'-x)$. (We suppress the free chion part of the Lagrangian.)

The QFTs based on (3) and (9) are equivalent in that, in conventional covariant perturbation theory, they lead to the same invariant matrix elements in various order of perturbation theory. The difference is that, in the formulation based on (9), the interaction term that contains the propagator leads to Feynman diagrams involving virtual chions, while the term that contains χ_0 correspond to diagrams that cannot be generated using the term with $D(x-x')$, such as those with external (physical) chion lines.

The Hamiltonian density corresponding to the Lagrangian (9) is given by

$$\mathcal{H}(x) = \mathcal{H}_\phi(x) + \mathcal{H}_\chi(x) + \mathcal{H}_{I_1}(x) + \mathcal{H}_{I_2}(x), \tag{10}$$

where

$$\mathcal{H}_\phi(x) = \dot{\phi}^*(x)\dot{\phi}(x) + \nabla\phi^*(x) \cdot \nabla\phi(x) + m^2\, \phi^*(x)\phi(x) \tag{11}$$

$$\mathcal{H}_\chi(x) = \frac{1}{2}\dot{\chi}_0^2 + \frac{1}{2}(\nabla\chi_0)^2 + \frac{1}{2}\mu^2\chi_0^2, \tag{12}$$

$$\mathcal{H}_{I_1}(x) = g\, \phi^*(x)\phi(x)\chi_0(x), \tag{13}$$

$$\mathcal{H}_{I_2}(x) = -\frac{g^2}{2} \int dx'\, \phi^*(x)\phi(x) D(x-x') \phi^*(x')\phi(x'), \tag{14}$$

and

$$D(x-x') = \int \frac{dk}{(2\pi)^{N+1}} e^{-ik\cdot(x-x')} \frac{1}{\mu^2 - k\cdot k}, \tag{15}$$

where $dk = d^{N+1}k$ and $k \cdot k = k^2 = k^\nu k_\nu$.

To specify our notation, we quote the usual decomposition of the fields in $N+1$ dimension:

$$\phi(x) = \int d^N q\, [(2\pi)^N 2\omega_q]^{-\frac{1}{2}} [A(\mathbf{q}) e^{-iq\cdot x} + B^\dagger(\mathbf{q}) e^{iq\cdot x}] \tag{16}$$

$$\chi(x) = \int d^N p\, [(2\pi)^N 2\Omega_k]^{-\frac{1}{2}} [d(\mathbf{p}) e^{-ip\cdot x} + d^\dagger(\mathbf{p}) e^{ip\cdot x}] \tag{17}$$

where $\omega_p = (p^2+m^2)^{\frac{1}{2}}$, $\Omega_q = (q^2+\mu^2)^{\frac{1}{2}}$, $q\cdot x = q^\nu x_\nu$ and $q^\nu = (q^0 = \omega_q, \mathbf{q})$. The momentum-space operators $A^\dagger$, A, $B^\dagger$, B obey the usual commutation relations. The nonvanishing ones are

$$[A(\mathbf{p}), A^\dagger(\mathbf{q})] = [B(\mathbf{p}), B^\dagger(\mathbf{q})] = \delta^N(\mathbf{p}-\mathbf{q}) \tag{18}$$

$$[d(\mathbf{p}), d^\dagger(\mathbf{q})] = \delta^N(\mathbf{p}-\mathbf{q}) \tag{19}$$

The Hamiltonian operator, $\hat{H}(t) = \int d^N x\, \hat{\mathcal{H}}(x)$, of the QFTheory is expressed in terms of the particle and antiparticle creation and annihilation operators $A^\dagger$, A, $B^\dagger$, B in the usual way. These operators are then commuted so that they stand in normal order in the Hamiltonian (we are not interested in vacuum-energy questions in this work).

3. Ansatz for the particle-antiparticle system, variational equation and effective potentials.

The simplest ansatz that can be chosen for a particle-antiparticle ($\phi\overline{\phi}$) state is

$$\mid \psi_2\rangle = \int d^N p_1\, d^N p_2\, F(\mathbf{p}_1, \mathbf{p}_2) A^\dagger(\mathbf{p}_1) B^\dagger(\mathbf{p}_2) \mid 0\rangle, \tag{20}$$

where F is an adjustable function. We use this trial state to evaluate the matrix elements needed to implement the variational principle (2), namely

$$\langle\psi_2 \mid : \hat{H}_\phi - E : \mid \psi_2\rangle = \int d^N p_1\, d^N p_2\, F^*(\mathbf{p}_1, \mathbf{p}_2) F(\mathbf{p}_1, \mathbf{p}_2) \Big[\omega_{p_1} + \omega_{p_2} - E\Big], \tag{21}$$

and

$$\begin{aligned}
\langle\psi_2 \mid : \hat{H}_I : \mid \psi_2\rangle &= \langle\psi_2 \mid : \hat{H}_{I_2} : \mid \psi_2\rangle \\
&= -\frac{g^2}{8(2\pi)^N} \int d^N p_1\, d^N p_2\, d^N p_1'\, d^N p_2'\, F^*(\mathbf{p}_1', \mathbf{p}_2') F(\mathbf{p}_1, \mathbf{p}_2) \\
&\quad \times \delta^N(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_1' - \mathbf{p}_2')\, e^{-i(\omega_{p_1}+\omega_{p_2}-\omega_{p_1'}-\omega_{p_2'})t} \frac{1}{\sqrt{\omega_{p_1}\omega_{p_2}\omega_{p_1'}\omega_{p_2'}}} \\
&\quad \times \Big[\frac{1}{\mu^2 - (p_1+p_2)^2} + \frac{1}{\mu^2 - (p_1'+p_2')^2} + \frac{1}{\mu^2 - (p_1'+p_1)^2} + \frac{1}{\mu^2 - (p_2'+p_2)^2}\Big].
\end{aligned} \tag{22}$$

We have normal-ordered the entire Hamiltonian since, at the present level of approximation (cf. the trial state (20)), this circumvents the need for mass renormalization which would otherwise arise in eq. (22). Also, in the Schrödinger picture we can take $t = 0$, and we do so henceforth.

If we now specialize to the rest frame, where $F(\mathbf{p}_1, \mathbf{p}_2) = f(\mathbf{p}_1)\delta^N(\mathbf{p}_1 + \mathbf{p}_2)$, then the variational principle (2) leads to the following momentum-space wave equation for the relative motion of the particle-antiparticle system:

$$\Big[2\omega_p - E\Big] f(\mathbf{p}) = \frac{g^2}{4(2\pi)^N} \int d^N p' f(\mathbf{p}') \frac{1}{\omega_p \omega_{p'}} \left[\frac{1}{\mu^2 + (\mathbf{p}' - \mathbf{p})^2 - (\omega_p - \omega_{p'})^2} - \frac{1}{4\omega_p^2 - \mu^2} \right] \tag{23}$$

Note that the kernel (momentum-space potential) in this equation contains two terms. The first corresponds to one-chion exchange and the second corresponds to virtual annihilation (this is perhaps more obvious from the four manifestly covariant terms in eq. (22)).

In the nonrelativistic limit, $\dfrac{p^2}{m^2} \ll 1$, this equation reduces to

$$\Big[\frac{\mathbf{p}^2}{m} - \epsilon\Big] f(\mathbf{p}) = \frac{g^2}{4(2\pi)^N m^2} \int d^N p' \, f(\mathbf{p}') \Big[\frac{1}{\mu^2 + (\mathbf{p}' - \mathbf{p})^2} - \frac{1}{4m^2 - \mu^2} \Big], \tag{24}$$

where $\epsilon = E - 2m$. In coordinate space, equation (24) is just the usual time-independent Schrödinger equation for the relative motion of the particle-antiparticle system:

$$-\frac{\hbar^2}{m} \nabla^2 \psi(\mathbf{r}) + V(r)\psi(\mathbf{r}) = \epsilon \psi(\mathbf{r}). \tag{25}$$

The potential $V(r)$ is a sum of an attractive Yukawa potential (due to one-chion exchange) and an repulsive (if $\mu < 2m$) contact potential (due to virtual annihilation). In 3+1 dimensions these are, explicitly,

$$V(r) = -\alpha \frac{e^{-\mu r}}{r} + \frac{4\pi\alpha}{4m^2 - \mu^2} \delta^3(\mathbf{r}), \tag{26}$$

where $\alpha = \dfrac{g^2}{16\pi m^2}$ is the effective dimensionless coupling constant.

It is clear from Eq. (23) that the relativistic, momentum-space one-quantum exchange potential is always attractive, while the virtual annihilation potential is repulsive if the mass μ of the mediating-field quantum is not too large, namely if $\mu \leq 2m$. However, if $\mu > 2m$ then the annihilation potential becomes attractive at low momenta.

4. Analytic solution of the variational particle-antiparticle equation with virtual annihilation interaction only.

The variational particle-antiparticle equation (23) cannot be solved analytically. Of course, numerical solutions can be obtained, and these will be discussed in section 6. In addition, for

the massless-exchange case, analytic perturbative solutions can be worked out, and these will be presented in section 5. However, if only the annihilation interaction is kept (i.e. the chion-exchange interaction is turned off), exact analytic solutions of such a simplified particle-antiparticle equation can be obtained (for both bound and scattering states). This is of interest as a solvable relativistic two-body equation, if for no other reason. Thus, if we neglect the first interaction term (the one-quantum-exchange term) in eq. (23) we obtain the rest-frame "annihilation-interaction" equation

$$[2\omega_p - E]f(\mathbf{p}) = f_0(\mathbf{p}) + \frac{g^2}{4(2\pi)^N}\int d^N p' f(\mathbf{p}')\frac{1}{\omega_p\omega_{p'}}\frac{1}{\mu^2 - 4\omega_p^2}, \tag{27}$$

where $f_0(\mathbf{p})$ is a "plane wave" solution of eq. (27) (with $g = 0$) representing the particle and antiparticle incident on each other with energy $E = 2\omega_{p_0}$. Of course, $f_0(\mathbf{p}) = 0$ for bound states. Because the annihilation interaction is entirely repulsive if $\mu \leq 2m$, we have only particle-antiparticle scattering solutions in that case.

We shall discuss the solution of eq. (27) in some detail only in 3+1 dimensions. It is evident that the integral on the right-hand-side of eq. (27) vanishes, except in S-states, hence is sufficient to write down the S-wave component of eq. (27), namely

$$f(p) = \frac{1}{4\pi p^2}\left[\delta(p - p_0) - \frac{\alpha\, m^2 A\, p^2}{(\omega_p - \omega_{p_0})(\omega_p^2 - (\frac{\mu}{2})^2)\,\omega_p}\right], \tag{28}$$

where

$$A = \int_0^\infty dp\, p^2 \frac{1}{\omega_p} f(p). \tag{29}$$

From equation (28) it follows that the S-wave phase shift η is given by

$$\tan\eta = -\pi\alpha\, m^2 A \frac{p_0}{\omega_{p_0}^2 - (\mu/2)^2}. \tag{30}$$

Substitution of eq. (28) into eq. (29) yields the result

$$A = \frac{1}{\omega_{p_0}\,(4\pi + \alpha\, I(m, \mu, p_0))}, \tag{31}$$

where

$$I(m, \mu, p_0) = m^2\,\mathcal{P}\int_0^\infty dp \frac{p^2}{(\omega_p - \omega_{p_0})\,\omega_p^2\,\left(\omega_p^2 - (\mu/2)^2\right)}. \tag{32}$$

This principal value integral is, explicitly,

$$\begin{aligned} I(b, q) = \quad & \frac{1}{2(q^2 + b^2)}\left\{\frac{b}{\sqrt{1 - b^2}}\ \left[\pi - 2\tan^{-1}\left(\frac{b}{\sqrt{1 - b^2}}\right)\right]\right. \\ & \left. - \frac{q}{\sqrt{1 + q^2}}\left[2\tanh^{-1}\left(\frac{q}{\sqrt{1 + q^2}}\right) + \frac{\pi\, q}{b + 1} - \frac{\pi\, b}{q(b + 1)}\right]\right\} \end{aligned} \tag{33}$$

if $0 < b < 1$, where $b^2 = 1 - (\mu/2m)^2$, $q = p_0/m$, and

$$I(q) = \frac{1}{q^2+1} + \frac{1}{(q^2+1)^{3/2}}\Big[\frac{\pi}{4}(1-q^2) - q\,\tanh^{-1}\left(\frac{q}{\sqrt{1+q^2}}\right)\Big] \tag{34}$$

for the massless exchange case, $\mu = 0$ (i.e. $b = 1$). With A as given in eq. (34), the tangent of the S-wave phase shift becomes

$$\tan\eta = -\frac{\pi\alpha\, m^2\, p_0}{\omega_{p_0}\left(\omega_{p_0}^2 - (\mu/2)^2\right)(4\pi + \alpha\, I(m,\mu,p_0))}, \tag{35}$$

from which the elastic particle-antiparticle scattering cross section $\sigma = \dfrac{4\pi}{{p_0}^2}\sin^2\eta$ is readily calculated.

For the massless chion exchange case ($\mu = 0$), the cross section, in units of π/m^2, starts from a value of $\dfrac{64\alpha^2}{(16 + [1 + 4/\pi]\alpha)^2}$ at $q = p_0/m = 0$, then decreases monotonically with increasing q to the asymptotic form $\dfrac{\alpha^2}{4q^6} \to 0$ as $q \to \infty$. Note that the maximum value of the cross section (which occurs at $q = 0$ for all α in this massless-exchange case) increases uniformly from zero at $\alpha = 0$ to an asymptotic value of $64\pi^2/(\pi+4)^2 = 12.3848$ as $\alpha \to \infty$.

When the mediating field quanta are massive then, for given α, the cross-section, as a function of collision energy, behaves qualitatively in a similar way to the massless case if $\mu/m < 1$. However the shape of the cross section changes as μ increases towards $2m$, in that it initially increases with the collision energy, reaches a maximum and then decreases towards zero as

$$\frac{\sigma}{\pi/m^2} \sim \frac{1}{4}\frac{\alpha^2}{q^6} + \frac{1}{16}\frac{\alpha^3}{(b+1)q^7} + O(\frac{1}{q^8}). \tag{36}$$

We shall not delve into a detailed discussion of the behaviour of $\sigma(q)$ for various α and b since the specific results can always be evaluated using the given analytic formulae (33 - 35).

It is of interest to note that in the non-relativistic limit, i.e. $p \ll m$, eq. (27) becomes,

$$\left[\frac{p^2}{m} - \epsilon\right] f(\mathbf{p}) = f_0(\mathbf{p}) + \frac{g^2}{4(2\pi)^N}\frac{1}{m^2(\mu^2 - 4m^2)}\int d^N p' f(\mathbf{p}'), \tag{37}$$

where $\epsilon = E - 2m$. The resulting I integral (cf. eq. (32)) diverges in $N = 2$ and 3 dimensions, thus resulting in a vanishing phase shift and cross section. This just reflects the "trivial" nature of the scattering by a repulsive delta function potential in non-relativistic (Schrödinger) theory in $N > 1$ dimensions ("trivial" in the sense that the S-matrix is unity).

For very massive mediating fields, $\mu > 2m$, the annihilation interaction becomes attractive at low momenta, and this leads to binding of the particle-antiparticle system if the coupling constant is large enough. The energy eigenvalue condition is (from eq. (27) with $f_0 = 0$)

$$1 = \frac{g^2}{4(2\pi)^N} \int d^N p \, \frac{1}{(2\omega_p - E)\, \omega_p^2 \, (\mu^2 - 4\omega_p^2)}. \tag{38}$$

In $N = 3$ dimensions equation (38) yields the result

$$\begin{aligned} \frac{2\pi}{\alpha} &= \frac{2}{4 + 4b^2 - \varepsilon^2} \left\{ \frac{b}{\sqrt{1+b^2}} \tanh^{-1}\left(\frac{b}{\sqrt{1+b^2}}\right) + \frac{\sqrt{4-\varepsilon^2}}{\varepsilon} \tan^{-1}\left(\frac{\sqrt{4-\varepsilon^2}}{\varepsilon}\right) \right. \\ &\left. + \pi \frac{b^2}{1+b^2} \frac{\varepsilon}{\sqrt{4-\varepsilon^2}} \right\} + \frac{\pi}{2} \frac{1}{1+b^2} \frac{1}{\varepsilon} \left(1 - \frac{4}{\sqrt{4-\varepsilon^2}}\right), \end{aligned} \tag{39}$$

where, now, $b^2 = (\mu/2m)^2 - 1$ and $\varepsilon = E/m$. A plot of $E(\alpha)$ for a representative value of $b = 1$, i.e. $\mu = 2\sqrt{2}\, m$, is given in Figure 1 (actually, $\alpha(E)$ is plotted). Note that binding does not set in until the coupling constant α exceeds a minimum value α_0. Thereafter the energy (i.e. particle-antiparticle mass) decreases monotonically with increasing α to an asymptotic value $E/m = \varepsilon_{\min}$, where both α_0 and $\varepsilon_{\min}$ vary with μ/m. The general expression for α_0 is

$$\alpha_0 = 2\pi \left[\frac{1}{2} \frac{1}{b\sqrt{b^2+1}} \tanh^{-1}\left(\frac{b}{\sqrt{b^2+1}}\right) + \frac{\pi}{4} \frac{1}{b^2+1} \right]^{-1} \sim \frac{8\pi}{\pi+2} + \frac{8\pi(4+3\pi)}{3(\pi+2)^2} b^2 + O(b^4). \tag{40}$$

The minimum value of α at which binding sets in (vis. $\alpha_0 = \frac{8\pi}{\pi+2} = 4.88812$) occurs when $b = 0$ (i.e. when μ just passes $2m$), but the binding is very weak, since $\varepsilon_{\min}$ is barely below 2 in that case. As b increases, so does α_0 and so does the binding energy. For α near α_0 the behaviour of $E(\alpha)$ is of the form

$$\frac{E}{m} = 2 - \left(\frac{4b^2}{\alpha_0}\right)^2 (\alpha - \alpha_0)^2 + O((\alpha - \alpha_0)^3). \tag{41}$$

Once again, it is of interest to note that the annihilation interaction, in $N = 2, 3$ spatial dimensions, can support bound states only in the relativistic formulation. This is because, had we started from the non-relativistic form of eq. (37) (with $f_0 = 0$ for bound states), then the eigenvalue equation corresponding to equation (38) would be

$$1 = \frac{g^2}{4(2\pi)^N} \frac{1}{m^2(\mu^2 - 4m^2)} \int d^N p \, \frac{1}{(p^2/m - \epsilon)}, \tag{42}$$

The integral (42) converges only for $N = 1$, whereas in the relativistic case the integral (38) converges for $N = 1, 2, 3$. In other words the short-range annihilation interaction does not support bound states non-relativistically in 3 or even 2 spatial dimensions (the interaction is an attractive delta-function potential in the non-relativistic limit), but it does support bound states

relativistically. This implies that the virtual annihilation interaction strengthens (relative to the non-relativistic delta function potential) if relativity is taken into account.

5. Perturbative results in 3+1 dimensions for the particle-antiparticle binding energy in the massless-exchange case.

The relativistic two-particle equation (23) can be reduced to radial form by setting

$$f(\mathbf{p}) = f(p)Y_{\ell m}(\hat{\mathbf{p}}), \tag{43}$$

where $p = |\mathbf{p}|$ and $Y_{\ell m}(\hat{\mathbf{p}})$ are the usual spherical harmonics, and carrying out the angular integration. The result, in $N = 3$ dimensions, is

$$\Big[2\omega_p - E\Big] f(p) = \frac{\alpha}{\pi}\int_0^\infty dp' \frac{p'}{p} f(p') K_\ell(p', p), \tag{44}$$

with

$$K_\ell(p', p) = \frac{m^2}{\omega_p \omega_{p'}}\left[Q_\ell(z) - \frac{\pi}{\omega_p^2 - (\frac{\mu}{2})^2} p\, p'\, \delta_{\ell 0}\right], \tag{45}$$

where

$$z = \frac{p^2 + p'^2 + \mu^2 - (\omega_p - \omega_{p'})^2}{2pp'}, \tag{46}$$

$\alpha = \dfrac{g^2}{16\pi m^2}$, and $Q_\ell(z)$ is the Legendre function of the second kind. This equation is similar to that derived by Di Leo and Darewych using a variational-perturbative approach [33]. That previous result corresponds to eq. (45) without the virtual-annihilation interaction (the second term in eq. (45)), and with $\omega_p = \omega_{p'}$ in the first, one-chion exchange term of eq. (45).

Since the solutions of equation (44) in the non-relativistic limit (without the virtual annihilation interaction) are the well known hydrogenic wave-functions in momentum-space [34], we can use them to obtain perturbative expressions to the particle-antiparticle mass (rest-energy). The result is

$$\begin{aligned} E_{n\ell}(\alpha) &= 2m - \frac{1}{4}m\alpha^2\frac{1}{n^2} - \frac{1}{16}m\alpha^4\left[\frac{2}{(2\ell+1)n^3} - \frac{3}{4n^4}\right] \\ &\quad + \frac{1}{8}m\alpha^4\left[\frac{4}{(2\ell+1)n^3} - \frac{1}{n^4}\right] + \Delta E_{anni} + O(\alpha^5). \end{aligned} \tag{47}$$

The terms on the right are the rest energy, the non-relativistic Balmer term, the O(α^4) correction to the kinetic energy, one-chion exchange interaction and the virtual annihilation interaction,

respectively. The correction due to the annihilation interaction is

$$\begin{aligned}\Delta E_{anni} &= \frac{\alpha}{\pi} m^2 \int_0^\infty \int_0^\infty dp\, dp' f(p) f(p') \frac{1}{2\omega_p^3 \omega_{p'}} p^2\, p'^2\, \delta_{\ell 0} \\ &= \frac{1}{8} m\alpha^4 \frac{1}{n^3} \delta_{\ell 0} + O(\alpha^5) \end{aligned} \tag{48}$$

The result (47) agrees with the earlier work [33], except that the annihilation correction (48) is new. The virtual annihilation correction is reminiscent of what is obtained for triplet $\ell = 0$ states of positronium, where one obtains $\frac{1}{4} m\alpha^4 \frac{1}{n^3} \delta_{\ell 0} \delta_{S1}$, with $\alpha = e^2/\hbar c$ in that case. Note that the "retardation term", $(\omega_p - \omega_{p'})^2$ of Eqs. (23) and (46), has no effect at $O(\leq \alpha^4)$.

As is well known, the massless Wick-Cutkosky model has been solved in the ladder approximation of the Bethe-Salpeter formulation [17, 18], as well as the light-cone ladder approximation [19, 20]. The expansion of these solutions in powers of α is found to be [19]

$$E/m = 2 - \frac{\alpha^2}{4n^2} - \frac{\alpha^3 \ln \alpha}{\pi n^2} + O(\alpha^3) \tag{49}$$

This is quite different from our result (47) beyond the $O(\alpha^2)$ Balmer term. The unusual $O(\alpha^3 \ln \alpha)$ terms are an artefact of the *ladder* Bethe-Salpeter formulation, and allegedly do not arise if crossed-ladder diagrams are included [35]. A discussion of the origin of the $\alpha^3 \ln \alpha$ term is given by A. Amghar and B. Desplanques [36]. In any case, our results contain no such terms, and are much more like the corresponding results for positronium in this respect (i.e. that the lowest order relativistic corrections to the Balmer result are $O(\alpha^4)$).

6. Numerical solution for the $\mu/m = 0$ and $\mu/m = 0.15$ cases.

We have solved equation (44) approximately for the ground state in $N = 3$ spatial dimensions, using the variational method with the trial wave function $f(p) = \dfrac{\omega_p}{(p^2 + b^2)^n}$, where b is an adjustable parameter, determined by minimizing

$$E = \left[\int_0^\infty dp\, 2\omega_p\, p^2 f^2(p) - \frac{\alpha}{\pi} \int_0^\infty \int_0^\infty dp' dp\, p' p f(p') f(p) K_\ell(p', p) \right] \Bigg/ \int_0^\infty dp\, p^2 f^2(p), \tag{50}$$

with respect to b, for various n and given α. Although these variational results are only approximations to the "exact" (i.e. numerical) solutions of eq. (44), they are in fact reasonably close to the numerical ones for the entire range of values of α considered, as shown in a previous study [33]. A list of ground-state values of E/m in the massive-exchange case for the present model is given in Table 1, for $n = 2, 2.5, 3, 3.5$. Generally, the $n = 2$ values are lowest for $\alpha < 0.5$, where relativistic effects are not so pronounced, but the $n = 2.5$ and $n = 3$ values are lower at strong

coupling. However, the various values look quite similar on a graph, and in the figures we shall plot curves corresponding to a single value of n only, as this will be sufficient for comparison purposes.

An advantage of using the variational solution is that we have an analytic representation of the wave-function, and so can examine its behaviour as the coupling constant α changes. Thus, we note that the values of the "inverse Bohr radius" parameter b, for any given n, increase monotonically with increasing α. Also, for any given value of α, the parameter b increases as n increases.

A plot of $E(\alpha, \mu = 0)$ for the ground state obtained in this way is shown in Figure 2. We plot two versions of our results, namely with and without the virtual annihilation interaction (second term of eq. (45)) included. Note that the affect of virtual annihilation is substantial, and it increases with increasing α.

As mentioned previously, the scalar Yukawa (or Wick-Cutkosky) model has been studied by many authors in various formalisms and approximations. It is therefore of interest to compare some of them to our results. Thus Figure 2 also contains plots of the classic solutions of Wick [17] and Cutkosky [18] of the Bethe-Salpeter equation in the ladder approximation, as well as the analogous light-cone calculations of Ji and Furnstahl [20]. We also plot the results of Di Leo and Darewych [33], which correspond to the present results with $\omega_p = \omega_{p'}$ (i.e. no retardation). None of these results contain the virtual annihilation interaction (which is repulsive) and so they should be compared with the present results without virtual annihilation.

It is evident from Fig. 2 that our variational results predict stronger binding that either of the ladder Bethe-salpeter results, or the Di Leo results. (The difference between our no-virtual-annihilation results and the Di Leo results show the effect of the retardation term $(\omega_p - \omega_{p'})^2$ in the potential of eq. (23).) Indeed, our results predict stronger binding than ladder B-S, even if we include the repulsive virtual annihilation interaction. Numerical values corresponding to Figure 2 are listed in Table 2 for $n = 2$, along with the wave-function parameter b. Table 3 is a list of corresponding results for $n = 3$.

It has been shown recently [37] that exact two-body eigenstates can be written down for the QFTheoretic Hamiltonian $H_\phi + H_{I_2}$ (cf. eqs. (10, 14)), that is, for the present model without free chions, provided that an "empty" vacuum state $|\tilde{0}\rangle$, annihilated by both positive and negative energy components of the field operator $\phi(x)$, is used. The use of such an empty vacuum state results in a relativistic two-body scalar equation that has both positive and negative energy solutions. Such negative-energy solutions do not arise (and should not arise) in the conventional QFT treatment that uses a Dirac "filled-negative-energy-sea" vacuum state, including the present work. Nevertheless the two-body equation obtained in [37] can be solved analytically [37, 38] in the massless-mediating-field case, and the positive energy, '$E \simeq 2m$' - like solution is

$$E = m\sqrt{2\left(1+\sqrt{1-\left(\frac{\alpha}{n}\right)^2}\right)} = m\left(2 - \frac{1}{4}\left(\frac{\alpha}{n}\right)^2 - \frac{5}{64}\left(\frac{\alpha}{n}\right)^4 + O(\alpha^6)\right) . \tag{51}$$

This result is quite different from that of the present treatment, or the ladder Bethe-Salpeter calculations. For one thing there is a rather low critical value of the coupling constant, namely

$\alpha_c = n$ ($\alpha_c = 1$ for the ground state) beyond which the two-body energy (rest mass) ceases to be real ($E/m = \sqrt{2} = 1.414...$ when $\alpha = \alpha_c$). This is similar to what happens for one-body Klein-Gordon or Dirac equations in a Coulomb potential. We also plot the result (51) in Fig. 2, and note that in the domain $0 \leq \alpha \leq 1$ it predicts stronger binding than any of the other results shown in the figure.

(An aside: The formula (51), and its $m_1 \neq m_2$ generalization derived in ref. [38], was obtained previously by Todorov [25] using a quasipotential approach. Todorov's article [25], which came to our attention recently, also contains a useful historical overview of earlier work on the relativistic two-body problem in QFT, including references to various "rediscovered" formulations and results.)

Figure 3 is a plot of the particle-antiparticle ground state energy in $N = 3$ spatial dimensions for the massive exchange case, with $\mu/m = 0.15$. Once again, we plot the solutions of our variationally derived equation (44) with and without virtual annihilation included. In addition, we plot some Bethe-Salpeter based quasipotential results that are given in a study of the $\phi^2\chi$ model by Nieuwenhuis and Tjon [27b], together with their numerical results obtained using a Feynman-Schwinger formulation. The various quasipotential results plotted in Figure 3 are explained in the paper of Nieuwenhuis and Tjon [27b], and this will not be repeated here. None of the calculations, save ours, contain the virtual annihilation interaction, so they should be compared to the version of our results without virtual annihilation. Some numerical values corresponding to the curves plotted in Figure 3 are listed in Table 4.

It is evident from Figure 3 that the ladder Bethe-Salpeter (BS) calculation predicts the weakest binding, while the Feynman-Schwinger (FS) results of Nieuwenhuis and Tjon predict the strongest binding. (The FS calculations are nonperturbative, but contain no loop effects.) The various quasipotential results are distributed between the ladder BS and the FS values, in the following order: Blankenbecler-Sugar-Logunov-Tavkhelidze [39], Gross (with retardation) [40] and "equal-time" [41]. All these quasipotential results are taken from Figure 1 of Nieuwenhuis and Tjon [27b]. Our present results (without virtual annihilation) fall above most of the quasipotential ones, and lie closest to the Blankenbecler-Sugar curve. We also include a plot of the two-body Klein-Gordon Feshbach-Villars formalism results of ref. [37], which lie very close to the Gross results. We can only speculate to what extent this is a coincidence (the Gross and FV formalism equations are not obviously similar).

Theussl and Desplanques [42] have recently calculated the two-body energy being studied here. They use the Bethe-Salpeter equation, but include crossed-ladder effects in an approximate perturbative way. Their results fall very close to the Gross quasipotential values (and to those of the Feshbach-Villars formulation) in the domain $\alpha < 1$. Therefore, we do not plot them in figure 3. The Theussl-Desplanques work underscores the inaccuracy of the ladder Bethe-Salpeter approximation and the importance of including crossed-ladder effects.

The fact that the present variational results show considerably weaker binding than the FS results of Nieuwenhuis and Tjon is perhaps not surprising, since the present calculation contains no "crossed-ladder" effects (the simple variational ansatz (20) is incapable of incorporating such effects). However, our results show considerably stronger binding than the ladder BS, even though

the latter uses essentially the same kernel (one-chion exchange) as the present calculation.

The quasipotential results differ from our variationally derived values in several respects. For one thing, the quasipotential equations are all different and somewhat ad-hoc (though physically motivated) one-time modifications of the BS equation, whereas our equations are obtained in a completely "ab-initio" way, and are limited primarily by the simple choice of ansatz (20) that we have made in this work. The quasipotential equations, in the unequal mass case, have the Klein-Gordon (KG) equation as their one-body limit. In this sense they are more like the two-body KG Feshbach-Villars results of refs. [37, 38], which also have the KG eq. as their one-body limit. The present variational approach leads to an equation, which, in the unequal mass case, does *not* have the KG one-body limit.

The stipulation that the "correct" one-body limit of a relativistic two-scalar-particle equation should be the KG equation is often made in the literature. However, this is a curious measure of correctness, since the KG equation has negative-energy solutions, which should not arise in a conventional QFT treatment that uses a Dirac "filled-negative-energy sea" vacuum. Indeed, negative energy solutions cannot, and should not, arise in the present calculations (we use the conventional Dirac vacuum), and so the unequal-mass counterpart of eq. (27) does not (and should not) have the KG equation as its one-body limit. On the other hand, the two-body KG Feshbach-Villars formalism equation (given in ref. [38] for the unequal mass case) is obtained using an "empty" vacuum, and so it does have negative-energy solutions, and also the KG one-body limit. Its binding energy predictions are similar to those of the quasipotential equations, even though it contains no retardation, or manifest crossed-ladder effects. The one-body limit equation in the present formalism has only positive-energy solutions, like the two-body equation (27).

Of course, we do not claim that our present results are "better" than the quasipotential equation ones (in the sense that they are closer to the unknown exact results for this model). The trial state (20) that we are using here is too simple to make any such claim. However, our approach is strictly variational, with nothing "put in by hand", and can be systematically improved by improving on the trial state (20) (which we are in the process of doing). It may be that the numerical FS results of Nieuwenhuis and Tjon are the most accurate binding-energy results available to date for the scalar Yukawa model, so it will be interesting to see how improved variational results will compare with this benchmark.

We should point out that the trial state (20) that has been used in this work is insensitive to the H_{I_1} term of the Hamiltonian (cf. Eq. (13)). Thus, it is suitable for describing stable particle-antiparticle states only, without explicit annihilation of the particle-antiparticle system, or decay of the excited bound states, with the emission of *physical* chions. It is possible to include such processes in the present formalism in various ways, such as perturbatively, or by suitable modification of the ansatz (20) (see, for example, refs. [33], [43]). However, we do not consider such processes in this paper.

7. Concluding remarks.

We have applied the variational method to the study of particle-antiparticle bound states in the scalar Yukawa (Wick-Cutkosky) model (scalar particles interacting via a massive or massless mediating scalar "chion" field). We have used a reformulated version of this theory in which a covariant Green function is used to eliminate the chion field partially, so that the chion propagator appears directly in the QFTheoretic hamiltonian.

A simple Fock-space trial function is used in the variational method. It leads to a relativistic particle-antiparticle momentum-space equation with the covariant one-chion exchange and virtual annihilation Feynman amplitudes appearing in the kernel (momentum-space potential) of the equation. The virtual annihilation interaction is repulsive, except if the mediating-field quantum (the chion) is very heavy (more massive than the combined rest mass of the particle and antiparticle) whereupon it becomes attractive at sufficiently large momenta. This particle-antiparticle equation has no negative energy solutions, i.e. it is free of any negative-energy "pathologies".

When the one-chion exchange interaction is turned off, we find that the resulting model theory (with a purely virtual-annihilation interaction) is analytically solvable for scattering states, and also for bound states when the chions are very massive. The virtual annihilation interaction reduces to a delta-function (contact) potential in the nonrelativistic limit. It supports no bound states in $N = 3$ spatial dimensions and the S-matrix is unity in this limit. However, if the relativistic equations are used, we find that bound states are possible and the scattering is not trivial. This analytically solvable relativistic model is instructive in understanding the effects of a relativistic generalization of the delta-function potential, such as occurs in the virtual annihilation interaction.

In the general case, with both one-chion exchange and virtual annihilation interactions included, the relativistic particle-antiparticle equation cannot be solved analytically (at least, we do not know how to do so). However, analytic expressions for the energy (rest mass) of the bound particle-antiparticle system can be obtained as an expansion in the effective dimensionless coupling constant α for the case of a massless mediating field. We give such an expression for the ground and arbitrary excited states of the system to $\mathrm{O}(\alpha^4)$ inclusive. We find that the lowest-order relativistic corrections to the Balmer formula are $\mathrm{O}(\alpha^4)$, much like for positronium, and quite unlike the predictions of the Bethe-Salpeter equation in the ladder approximation, which include unusual $\alpha^3 \ln \alpha$ terms.

Lastly, we calculated the particle-antiparticle bound-state energy for arbitrary α in the ground state. This was done using a variational approximation rather than numerical integration of the two-body equation, as the results are not much different and the variational method allows one to exhibit the behaviour of the wave-function more transparently for various values of α. We study the case where the exchanged quantum has mass $\mu = 0$, and find that our results predict stronger binding than ladder Bethe-Salpeter approximations (Wick-Cutkosky or light-front solutions). For the case of massive chion exchange, with $\mu = 0.15$, we find analogous behaviour. However, our results show weaker binding than most of the quasipotential reductions of the Bethe-Salpeter equations.

The present approach has several attractive features. Firstly, it leads to equations with no

negative-energy or mixed-energy solutions, such as arise in many other formulations (though such negative-energy and mixed-energy solutions usually are not discussed). Secondly, the results are strictly variational, with no perturbative approximations. Thus the results are applicable at strong coupling, at least in principle (we hasten to add that variational results are only as good as the trial states employed in their use). Unlike the quasipotential reductions of the Bethe-Salpeter equation, our results are rigorous in the sense that nothing is put in by hand. Thirdly, the method is amenable to systematic improvement by improving the variational trial state. Lastly, the method is straightforwardly generalizable to relativistic three or more particle systems, though, of course, one is then faced with the usual complexity of a relativistic many-body problem [44].

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Figure captions

Figure 1

Two-particle bound-state energy E (eq. 39) due to virtual-annihilation interaction only, for various values of the coupling constant α, at a mediating-quantum mass of $\mu = 2\sqrt{2}m$ (i.e. $b = 1$). Binding sets in for $\alpha > \alpha_0 = 8.9203$, and $E/m \to \epsilon_{\min} = 1.5058$ for $\mu/m = 2\sqrt{2}$.

Figure 2

Particle-antiparticle bound state mass E/m for various values of the coupling constant α, massless quantum exchange. Curves, from highest to lowest: ligh-front ladder Bethe-Salpeter [Ji and Funstahl, 20a]; ladder Bethe-Salpeter [Wick and Cutkosky, 17, 18]; equations (44) without virtual annihilation or retardation [Di Leo and Darewych, 33]; equation (44) with virtual annihilation and retardation ($n = 2$); equation (44) with retardation but without virtual annihilation ($n = 2$); Feshbach-Villars formalism [Darewych, 37]. Note that all these results, save one, do not contain the virtual-annihilation interaction.

Figure 3

Particle-antiparticle bound state mass E/m for various values of the coupling constant α, massive quantum exchange: $\mu/m = 0.15$. Curves, from highest to lowest: ladder Bethe-Salpeter [Wick and Cutkosky, 17, 18]; present results of equation (44) with virtual annihilation ($n = 2$); equation (44) without virtual annihilation ($n = 2$); crosses: Blankenbecler-Sugar [27b, 39]; Feshbach-Villars formalism [Darewych, 37]; diamonds: Gross [27b, 40]; boxes: equal-time [27b, 41]; circles: Feynam-Schwinger formalism [27b]. Note that all these results, save one, do not contain the virtual-annihilation interaction.

Table 1.

Ground state particle-antiparticle mass (rest energy) E_2/m obtained from eq. (50) using $f(p) = \omega_p/(p^2+b^2)^n$ for various values of the coupling constant $\alpha = \frac{g^2}{16\pi m^2}$ and variational parameter n. Massive exchange case, $\mu/m = 0.15$. Virtual annihilation is included here. The number in brackets below each energy is the corresponding value of b/m.

	$n=2$	$n=5/2$	$n=3$	$n=7/2$
α	E_2/m (b/m)	E_2/m (b/m)	E_2/m (b/m)	E_2/m (b/m)
0.3	2.0	2.0	2.0	2.0
	1.991393	1.991776	1.992553	1.993181
0.5	(0.159615)	(0.215695)	(0.255046)	(0.287139)
	1.972650	1.971980	1.973059	1.974105
0.7	(0.219966)	(0.301969)	(0.360828)	(0.409350)
	1.946141	1.943654	1.944858	1.946269
0.9	(0.267098)	(0.369155)	(0.443004)	(0.504074)
	1.930478	1.926874	1.928095	1.929676
1.0	(0.287458)	(0.398092)	(0.478350)	(0.544784)
	1.713297	1.694965	1.695824	1.699076
2.0	(0.430321)	(0.598907)	(0.722546)	(0.825342)
	1.428109	1.393489	1.394290	1.399622
3.0	(0.520198)	(0.722945)	(0.872253)	(0.996653)
	1.104266	1.054041	1.055524	1.063454
4.0	(0.585899)	(0.812489)	(0.979783)	(1.115435)
	0.755363	0.690629	0.693494	0.704603
5.0	(0.637454)	(0.882107)	(1.061143)	(1.201048)
	0.388913	0.310760	0.315679	0.330580
6.0	(0.679660)	(0.938688)	(1.122941)	(1.266242)

Table 2.

Value of E_2/m (ground state) for $\mu/m = 0$ in $3+1$, eq. (50) with $n = 2$.

α	With annihilation E_2/m (b/m)	Without annihilation E_2/m (b/m)	Di Leo [33] (no ann. or retard.) E_2/m (b/m)
0.1	1.9975 (0.0492)	1.9975 (0.0495)	1.997527 (0.049049)
0.3	1.9792 (0.1327)	1.9786 (0.1372)	1.979182 (0.132481)
0.5	1.9469 (0.1963)	1.9441 (0.2075)	1.947012 (0.195480)
0.7	1.9045 (0.2463)	1.8975 (0.2647)	1.904898 (0.244621)
0.9	1.8544 (0.2872)	1.8413 (0.3126)	1.855351 (0.284642)
1.0	1.8271 (0.3053)	1.8101 (0.3339)	1.828330 (0.302153)
2.0	1.4962 (0.4341)	1.4203 (0.4902)	1.503838 (0.425538)
3.0	1.1026 (0.5156)	0.9415 (0.5916)	1.121291 (0.501635)
4.0	0.6744 (0.5748)	0.4120 (0.6662)	0.707645 (0.555879)
5.0	0.2241 (0.6208)	-0.1500 (0.7248)	0.274678 (0.597410)

Table 3.

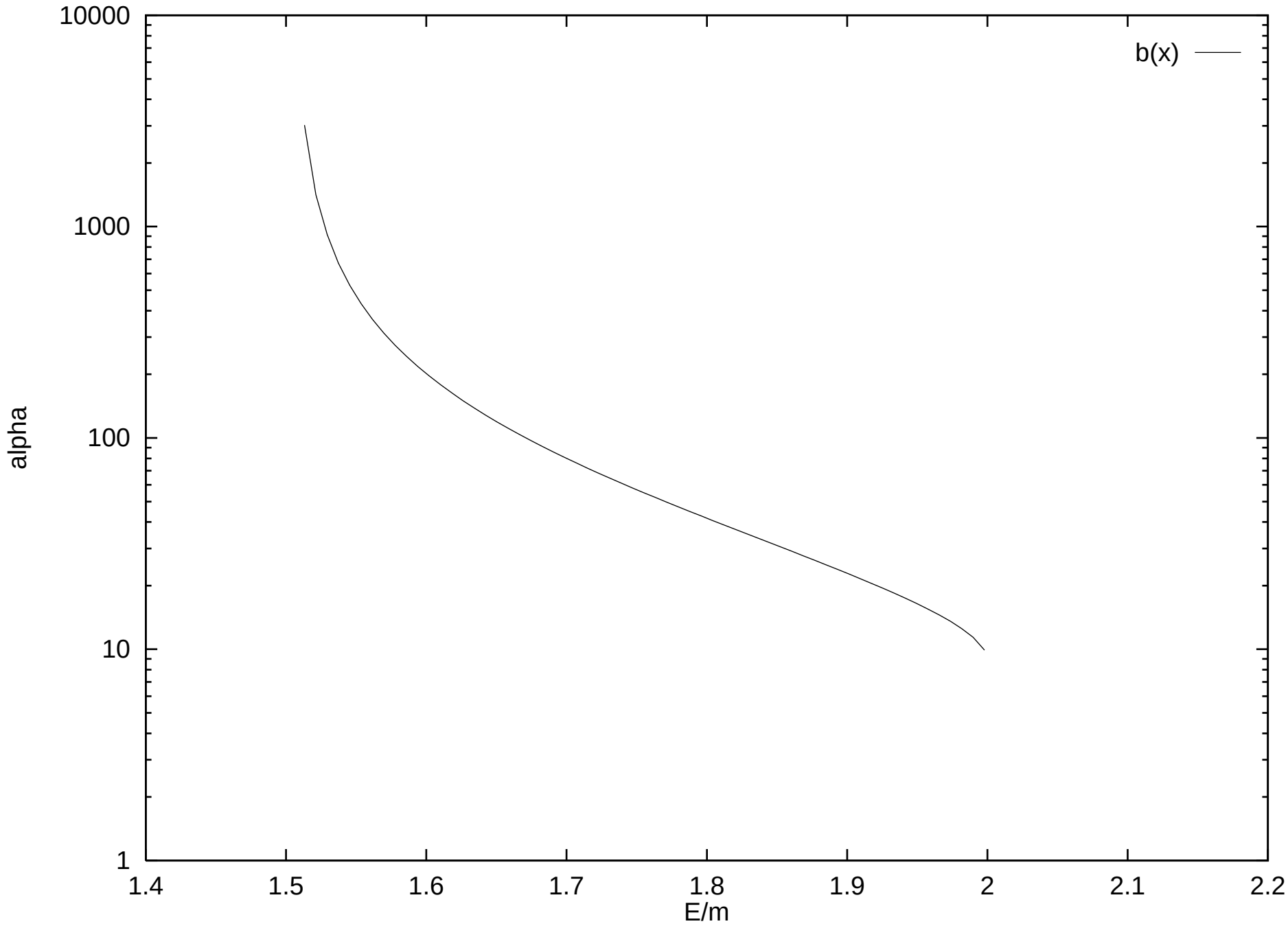
Value of E_2/m (ground state) for $\mu/m = 0$ in $3+1$, eq. (50) with $n = 3$.

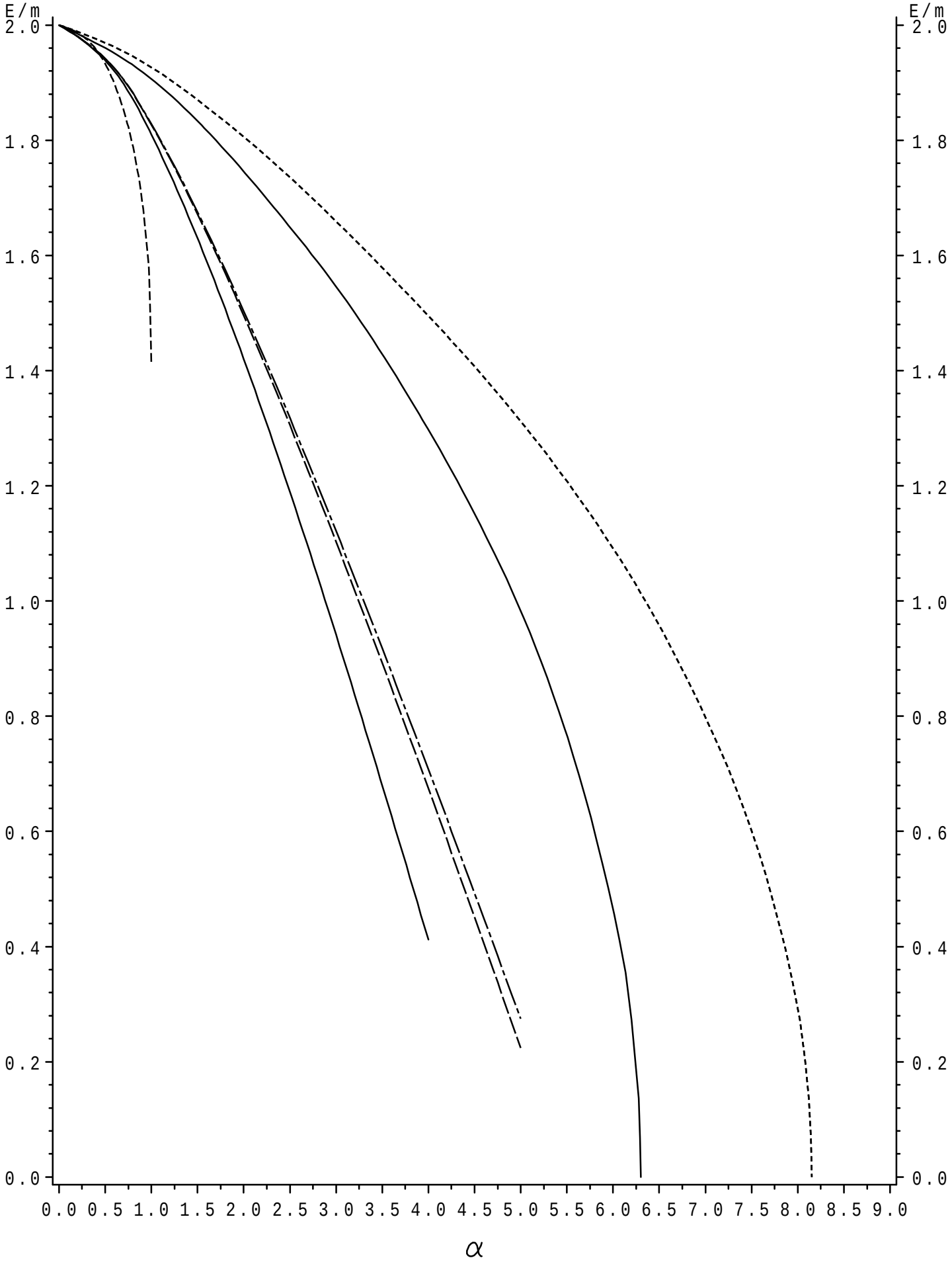
α	With annihilation E_2/m (b/m)	Without annihilation E_2/m (b/m)	Di Leo [33] (no ann. or retard.) E_2/m (b/m)
0.1	1.9976 (0.0744)	1.9976 (0.0747)	1.997605 (0.074211)
0.2	1.9906 (0.1443)	1.9905 (0.1461)	1.990593 (0.144197)
0.3	1.9794 (0.2079)	1.9790 (0.2128)	1.979502 (0.207603)
0.4	1.9645 (0.2647)	1.9636 (0.2738)	1.964567 (0.263998)
0.5	1.9465 (0.3153)	1.9445 (0.3292)	1.946607 (0.313968)
0.6	1.9257 (0.3606)	1.9224 (0.3796)	1.925973 (0.358409)
0.7	1.9036 (0.4013)	1.8974 (0.4255)	1.903044 (0.398195)
0.8	1.8774 (0.4383)	1.8699 (0.4674)	1.878128 (0.434075)
0.9	1.8505 (0.4720)	1.8403 (0.5060)	1.851481 (0.466668)
1.0	1.8219 (0.5030)	1.8086 (0.5417)	1.823314 (0.496469)

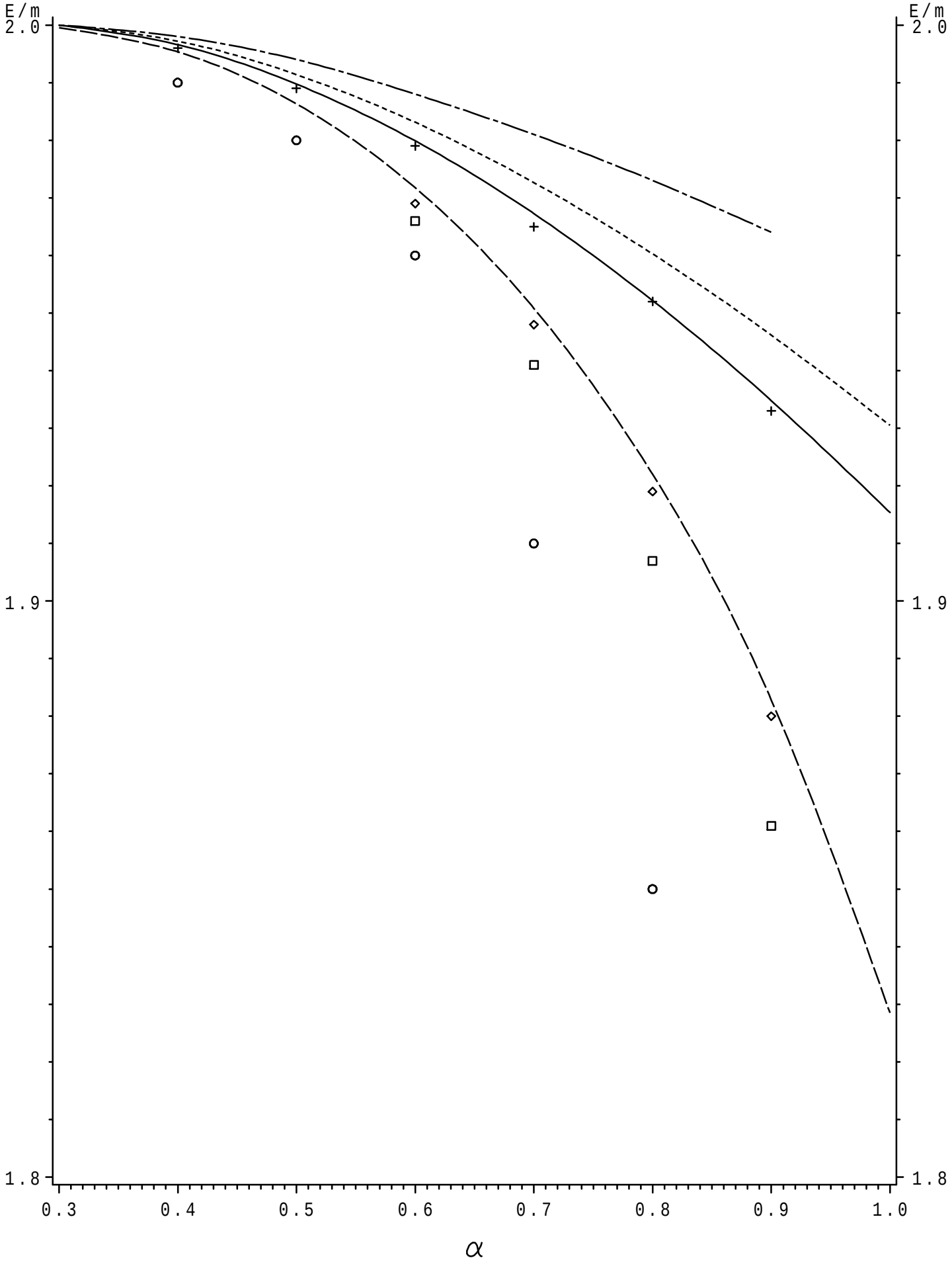
Table 4.

Value of E_2/m (ground state) for $\mu/m = 0.15$ in $3+1$. Results given in columns 2,3,4 and 6 were read off fig. 1 of [27b], hence the accuracy of the last figure is questionable.

α	Niewenhuis and Tjon	Equal Time	Gross eq. (with retard.)	Darewych [37]	Blankenbecler and Sugar	Di Leo [33]	Present results ($n = 2$) no annih.
0.3				1.999536		2.0	2.0
0.4	1.99			1.99534	1.996	1.996946	1.996582
0.5	1.98			1.98630	1.989	1.990833	1.989742
0.6	1.96	1.966	1.969	1.97176	1.979	1.982148	1.979868
0.7	1.91	1.941	1.948	1.95081	1.965	1.971207	1.967249
0.8	1.85	1.907	1.919	1.92199	1.952	1.985276	1.952143
0.9	1.77	1.861	1.880	1.88282	1.933	1.943577	1.934776
1.0				1.82847		1.927296	1.915346
2.0						1.703450	1.637352
3.0						1.412621	1.262094
4.0						1.084915	0.830372
5.0						0.733877	0.361942


10000
1000
100
10
1
alpha
1.4
1.5
1.6
1.7
1.8
1.9
2
2.1
2.2
E/m
b(x)


E/m
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0 0.5 1.0 1.5 2.0 2.5 3.0 3.5 4.0 4.5 5.0 5.5 6.0 6.5 7.0 7.5 8.0 8.5 9.0
α


E/m
2.0
1.9
1.8
E/m
2.0
1.9
1.8
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
α